Fundamentals of Algebraic geometry

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There is a good deal of algebraic geometry that I am encountering and it seems that the concepts are actually more intelligible than they first seem, they are just in a slightly different language. What I need is a dictionary. I will go through the first two chapters of [**?**].

The next thing is the highly abstract functor of points appraoch that I think I need to look at the details of before it will stop bothering me. Grothendieck in the introduction to EGA (I) gives a good overview of the basic peices and set up.

Contents

1 Prerequisites

Conventions are that all rings are commutative with identity. k denotes an algebraically closed field.

A non-empty subset $Y \subseteq X$ of a topological space is ***irriducible*** if it cannot be written as the union of two proper, closed and disjoint subsets.

The ***radical*** of an ideal $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} \quad := \quad \{f : \exists n \in \mathbb{N} \;\; f^n \in \mathfrak{a}\}$$

An ideal is radical if its equal to its own radical.

A topological space X is ***noetherian*** if it satisfies the ascending chain condition: For any sequence of closed sets $\cdots \subseteq Y_2 \subseteq Y_1$ there exists an integer r such that $Y_r = Y_{r+n}$ for all $n \in \mathbb{N}$. ie. the chain stabilises.

Note that when look at $\mathbb{A}^n$ then this is equivilent to the chain of ideals $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$ terminating, i.e. A being Noetherian (as an algebra).

In a noetherian topological space every non-empty closed subset can be expressed as a finite union of irrducible closed cubsets. If we require that the irreducibles do not contain each other then this decomposition becomes unique.

very clearly related to representations, make explicit, complete reducibility...

For the topological space X the ***dimension*** is defined as the supremum of the integers n such that there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irriducible closed subsets of X.

The dimension of $\mathbb{A}^n$ is n.

Theorem. *For an affine algebraic set Y*

$$dimY = dimA(Y)$$

The ***height*** of a prime ideal $\mathfrak{p}$ is the supremum of the integers such that there exists a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

The ***Krull dimension*** of A is the supremum of all heights of prime ideals of A.

A ***graded ring*** is a ring S with a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

A direct sum of abelian groups such that $S_d \cdot S_{d'} \subseteq S_{d+d'}$. An element of S_d is called a ***homogeneous element*** of degree d. Every element of S can be written uniquely as a finite sum of homogeneous elements.

An ideal $\mathfrak{a}$ is a ***homogeneous ideal*** if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$$

Lemma.

- *An ideal is homogeneous iff every elements homogeneous components are also in the ideal*
- *If $\mathfrak{a}, \mathfrak{b}$ are homogeneous ideals then their sum, product, intersection and radicals are also homogeneous.*
- *A homogeneous ideal is prime iff for any homogeneous elements whose product is in the ideal then one of the productands is in the ideal.*

2 Varieties

Affine

Define $A = k[x_1, ..., x_n]$ then if $T \subseteq A$ we can Define

$$Z(T) \quad := \quad \{P \in k^n : f(P) = 0 \ \ \forall f \in T\}$$

A subset $Y \subseteq k^n$ is ***algebraic*** if there is some $T \subseteq A$ such that $Y = Z(T)$.

Affine n space over k, denoted $\mathbb{A}^n_k$, is the topological space k^n with closed sets the collection of algebraic sets, called the Zariski topology.

An ***affine algebraic variety*** is an irreducible closed subset of $\mathbb{A}^n$ with the induced subspace topology. An oper subset of an affine variety is called ***quasi-affine variety***.

To a subset $Y \subseteq \mathbb{A}^n$ we can assign an ideal of A by

$$I(Y) \quad := \quad \{f \in A : f(P) = 0 \ \ \forall P \in Y\}$$

Notice that

- For any ideal $\mathfrak{a} \subseteq A$
$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$
- For any subset $Y \subseteq \mathbb{A}^n$
$$Z(I(Y)) = \bar{Y}$$
the closure.

Theorem (Nullstellensatz)**.** *If k is algebraically closed field, $\mathfrak{a}$ is an ideal in $A = k[x_1, ..., x_n]$ and $f \in A$ such that $f(Z(\mathfrak{a})) = 0$ then $f^r \in \mathfrak{a}$ for some $r > 0$*

Theorem. *There is a one to one mapping*

$$\{\textit{Algebraic Sets in } \mathbb{A}^n\} \leftrightarrow \{\textit{Radical ideals}\}$$

$$Y \mapsto I(Y)$$

$$Z(\mathfrak{a}) \leftarrow\!\shortmid \mathfrak{a}$$

Under this map an algebraic set is irreducible iff its corresponding ideal is prime.

If $Y \subseteq \mathbb{A}^n$ affine algebraic set the ***affine coordinate ring*** of Y is $A(Y) \quad := \quad A/I(Y)$.

Theorem. *The above map is a bijection between finitely generated k-algebras that are integral domains and coordinate rings of affine varieties.*

Theorem. *Evey algebraic set is the union of affine varieties.*

Projective

$\mathbb{P}^n_k$ ***projective n space over k*** is as a (topological) quotient of $\mathbb{A}^{n+1} - (0, ..., 0)$ by the relation that two points are equivilient if they lie on the same line through the origin.

If $S = k[x_0, ..., x_n]$ then we make S into a graded ring by taking S_d to be the span of monomials of total weight d. If f is a polynomial it doesnt necissarily give a well defined function on $\mathbb{P}^n$, if f is homogeneous of degree d then for any $\lambda \in k - 0$

$$f(\lambda a_0, ..., \lambda a_n) = \lambda^d f(a_0, ..., a_n)$$

Hence the property of f being zero is well defined on the equivilence classes of the points of $\mathbb{P}^n$.

Now we understand polynomials and can repeat the setup of the affine space. Namely we have the map from collections of homogeneous polynomials, say $T \subseteq S$, to $\mathbb{P}^n$ given by

$$Z(T) \quad := \quad \{P \in \mathbb{P}^n : f(P) = 0 \ \ \forall f \in T\}$$

If $\mathfrak{a}$ is a homogeneous ideal then $Z(\mathfrak{a}) \quad := \quad Z(\text{homogeneous elements of } \mathfrak{a})$.

We say that a subset of $Y \subseteq \mathbb{P}^n$ is ***algebraic*** if there exists a set of homogeneous polynomials $T \subseteq S$ such that $Y = Z(T)$. We use these algebraic sets again as the closed sets on $\mathbb{P}^n$ and call this the Zariski topology on $\mathbb{P}^n$.

A ***projective algebraic variety*** is an irreducible algebraic set in $\mathbb{P}^n$ with the subspace topology.

If $Y \subseteq \mathbb{P}^n$ then the homogeneous ideal associated is

$$I(Y) \quad := \quad \langle\{f \in S : f \text{ is homogeneous and } \forall P \in Y \quad f(P) = 0\}\rangle$$

If Y is an algebraic set then its ***homogenous coordinate ring*** is $S(Y) \quad := \quad S/I(Y)$.

Theorem. *$\mathbb{P}^n$ has an open cover by sets that are homeomorphic to $\mathbb{A}^n$.*

As a corrollery any projective variety admits a cover by opens that are homeomorphic to affine varieties.

2.1 Morphisms

Let Y be a quasi-affine (projective) variety in A^n ($\mathbb{P}^n$). A function $f : Y \to k$ is ***regular at a point*** $P \in Y$ if there is an open neighbourhood $P \in U \subseteq Y$ and (homogenous of the same degree) polynomials $g, h \in A$ (S) such that h is nevery zero on U and $f = g/h$. f is ***regular on Y*** if it is regular at every point. A regular function is continuous when k is identified with $\mathbb{A}^1_k$ with the Zariski topology.

Definition. *The category of varieties has*

- *Objects are: Affine, quasi-affine, projective and quasi-projective varieties. We call these simply* ***varieties*** *over k.*
- *Morphisms: A* ***morphism of varieties*** *is a continuous map $\varphi : X \to Y$ such that for every open $V \subseteq Y$ and every regular function $f : V \to k$ then $f \circ \varphi : \varphi^{-1}V \to k$ is regular.*

2.2 Rings of Functions

If Y is a variety then we denote $\mathcal{O}(Y)$ the ***ring of regular functions*** on Y (pointwise operations). If $P \in Y$ then the ***local ring of P on Y*** is the ring of germs of regular functions near P, we denote this $\mathcal{O}_{P,Y}$.

$$\mathcal{O}_{P,Y} = \{[(f, U)] | P \in U \subseteq Y \text{ open and } f : U \to k \text{ regular}\}$$

where $(f, U) \sim (g, V)$ iff $f|_{U \cap V} = g|_{U \cap V}$. To add and multiply the germs we simply add and multiply the functions pointwise and intersect the open sets. Note that k is a field with unit and so the two obvious units of these rings are

$$1 : x \mapsto 1 \in k$$

$$0 : x \mapsto 0 \in k$$

So these are rings. $\mathcal{O}_{P,Y}$ is local with unique maximal ideal given by

$$\mathfrak{m}_P = \{(f, U) : f(P) = 0\}$$

Moreover $\mathcal{O}_{P,Y}/\mathfrak{m} \cong k$.

We now define the ***function field*** of Y which we denote $K(Y)$ as the collection of germs at every (any) point. This is a field under the same operations as above.

Theorem. *$\mathcal{O}(Y), \mathcal{O}_{P,Y}$ and $K(Y)$ are invariants of the variety Y up to isomorphism. (If two varieties are iso then so are these rings).*

Theorem. *Let Y be an affine variety with coordinate ring $A(Y)$*

- *$\mathcal{O}(Y) \cong A(Y)$*
- *The map $P \mapsto \mathfrak{m}_P$ gives a bijection between points of Y and maximal ideals of $A(Y)$*
- *For every P we have that $\mathcal{O}_{P,Y} \cong \mathfrak{m}_P^{-1} A(Y)$ the localisation of $A(Y)$ at $\mathfrak{m}_P$*
- *$K(Y) \cong A(Y)/$*

quotient what????

Theorem. *Let Y be a projective variety with homogenous coordinate ring $S(Y)$*

- *$\mathcal{O}(Y) \cong k$*
- *For every P we have that $\mathcal{O}_{P,Y} \cong \mathfrak{m}_P^{-1} S(Y)$*
- *$K(Y) \cong (0)^{-1} S(Y)$*

Theorem. *K(Y) is a fintiely generated field extension of k.*

Theorem. *Let X be a variety and Y be an affine variety. There is a natural bijection*

$$\mathrm{Hom}_{var}(X, Y) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}_{k-alg}(A(Y), A(X))$$

It follows that two affine varieties are isomorphic iff their coordinate rings are.

2.3 The Rest

The rest of chapter two is dedicated to I guess more specifically "algebraic geometry" questions. Heartshorn comments that the key questions are

> ... the problem is to classify all algebraic varieties up to isomorphism The first part is to classify varieties up to birational equivalence. The second part is to identify a good subset of a birational equivalence class, such as the nonsingular projective varieties, and classify them up to isomorphism. The third part is to study how far an arbitrary variety is from one of the good ones considered above. In particular, we want to know (a) how much do you have to add to a nonprojective variety to get a projective variety, and (b) what is the structure of singularities, and how can they be resolved to give a nonsingular variety?

So there is much discussion of rational maps between varieties, singular vs non-singular varieties and how to go between them (blow up) and affine vs projective varieties and how to go between them.

3 Schemes

Let A be a ring. We associate to it the set $\mathrm{Spec}(A)$ which is the collection of prime ideals. If $\mathfrak{a} \subseteq A$ is an ideal then we define a subset $V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$ to be the collection of prime ideals containing $\mathfrak{a}$. $\mathrm{Spec}(A)$ is give the topology whose closed sets are generated by $V(\mathfrak{a})$.

If $\mathfrak{p} \in \mathrm{Spec}(A)$ a prime ideal then we denote the localisation of A at $\mathfrak{p}$ as $A_{\mathfrak{p}}$. We assign a sheaf to $\mathrm{Spec}(A)$, called the structure sheaf $\mathcal{O}$ defined by

$$\mathcal{O}(U) \quad := \quad \left\{ s : U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} | \forall \mathfrak{p} \in U \;\; s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and s is locally a quotient of elements of } A \right\}$$

To make the condition precise: For each $\mathfrak{p} \in U$ we require there to exist a neighbourhood of $\mathfrak{p} \in V \subseteq U$ and $a, f \in A$ such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$.

O(U) looks a lot like the adeles...

Let $f \in A$ we define $D(f)$ to be the compliment of $V((f))$. (the ideal generated by f). These open sets form a basis of the topology on $\mathrm{Spec}(A)$.

Theorem. *If A is a ring and $(\mathrm{Spec}(A), \mathcal{O})$ its spectrum then*

- *For every $\mathfrak{p} \in \mathrm{Spec}(A)$ the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$.*
- *For every $f \in A$ then $A_f \cong \mathcal{O}(D(f))$*
- *In particular $\mathcal{O}(\mathrm{Spec}(A)) \cong A$.*

We want to make this assignment functorial so we now construct the category of locally ringed spaces.

recall a morphism of sheaves of rings and pulling back....

Definition. *The **category of ringed spaces** is:*

- *Objects: A pair, $(X, \mathcal{O})$, consisting of a topological space X and a sheaf of rings.*
- *Morphisms: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ and a morphism of sheaves of rings $\varphi : \mathcal{O}_Y \to f_*\mathcal{O}_X$*

Definition. *The **category of locally ringed spaces** is:*

- *Objects: Locally ringed spaces; A ringed space $(X, \mathcal{O})$ such that all the stalks $\mathcal{O}_x$ are local rings.*

- *Morphisms: Local homomorphisms; A morphism of locally ringed spaces, $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, such that for every $P \in X$ the induced map on the stalks $\varphi_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings, i.e. the image of the maximal ideal lands in the other maximal ideal.*

Locally ringed spaces form a (non-full) subcategory of ringed spaces.

Lemma. Spec : *Rings* $\to$ *Locally Ringed Spaces sending* $A \mapsto (\mathrm{Spec}(A), \mathcal{O}_A)$ *defines a full functor.*

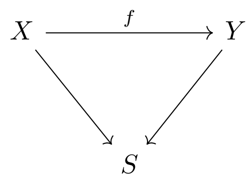
An ***Affine Scheme*** is a locally ringed space that is isomorphic as a locally ringed space to the spectrum of a ring. A ***Scheme*** is a locally ringed space $(X, \mathcal{O})$ such that every point has a neighbourhood U such that $(U, \mathcal{O}|_U)$ is an affine scheme. A morphism of schemes is a morphism of locally ringed spaces.

why did we go from the abstract varieties above to shchemes though? Arbitrary rings?

3.1 Varieties As Schemes

First we generalise affine space. Define for k a field (no longer algebraically closed) the scheme ***affine n space over k*** to be $\mathbb{A}^n_k := \mathrm{Spec}\, k[x_1, ..., x_n]$. The zero ideal corresponds to a generic point, its closure is the whole space. Points corresponding to maximal ideals are closed. If k happens to be algebraically closed the closed points are in bijection with the elements of k^n.

If we have a fixed scheme S then a ***scheme over*** S is a morphism of schemes $X \to S$. We are more interested in the category of schemes over a fixed scheme, so if we have two schemes over S, say $X, Y \to S$ then a morphism of schemes over S is a morphism of schmes $X \to Y$ such that the diagram commutes.



We denote the cateogory of schemes over S by $Sch(S)$ or if A is a ring we write $Sch(A)$ for $Sch(\mathrm{Spec}(A))$.

Theorem. *If k is algebraically closed then there is a fully faithful functor*

$$t : Var(k) \to Sch(k)$$

Moreover

- *The topological space of the variety is homoemorphic to the colleciton of closed points of the topological space assigned as the scheme*
- *The sheaf of regular functions on the variety is the restriction of the structure sheaf of the scheme via this homeomorphism*

Explicitly if X is a topological space then

$$t(X) := \{ \text{irreducible closed subsets of } X \}$$

with the topology generated by having closed sets subsetse of the form $t(Y)$ for some $Y \subseteq X$ closed (possibly empty). On maps $f : X \to Y$ is sent to

$$t(f) : t(X) \to t(Y)$$

$$t(f)(Q) = \overline{f(Q)}$$

We also need to give the sheaf of local rings: First define

$$\alpha : X \to t(X)$$

$$P \mapsto \overline{\{P\}}$$

All of this has been purely topological. If V is a variety over an algebraically closed field with sheaf of regular functions $\mathcal{O}_V$ then $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme.

3.2 Confused why this matters

A generic point is a point such that the closure of that point is the whole space. The zero ideal corresponds to a generic point? You have to add generic points to the topological space of a vaiety to get a scheme for some reason?

4 EGA: The Functor of Points

This is a summary / review / expansion of the introduction of EGA (I) (1971) as translated [?], where Grothendieck beautifully motivates the modern constructions in algebraic geometry.

4.1 History origins and insights

The goal is to recapitulate the history of mathematics and show how modern algebraic geometry is a natural development.

4.1.1 Algebra

At the begining of time people were interested in the solutions of systems of polynomial equations. Broadly speaking this was the focus of algebra. In modern mathematical language we would set up the problem that the ancients and moderns alike have been trying to solve:

Definition (Problem 1)**.** *Let k be a commutative ring with unit. Let $P_I = P := k[(T_i)_{i \in I}]$ be the polynomial ring in some number of variables, and for every $t = (t_i)_{i \in I} \in k^I$ denote the ring homomorphism $P \to k$*

$$ev_t : T_i \mapsto t_i$$

simply by $F \mapsto F(t)$.

Now given a collection of polynomials $(F_j)_{j \in J} \in P^J$ we want to find all the $t \in k^I$ such that for every $j \in J$

$$F_j(t) = 0$$

this is algGeo

the efforts to answer this question (a vague notion of total understanding) have proceeded often by asking other perhaps more tractable questions such as

Definition (Problem 2)**.**

- *Is the collection specified empty or non-empty*
- *Is it finite or infinite*
- *Can we calculate the number of elements explicitly, or give an asymptotic bound*

4.1.2 Geometry

The objects of geometry, are intimately connected with Problem 1. In essence much of classical geometry is the case where the ring is $\mathbb{R}$ and there are perhaps two or three variables i.e. polynomials in $\mathbb{R}[x, y]$. These are things like quadrics, conics, elliptic curves etc.

perhaps since Descarte

Studies of the questions such as Problem 2 over $\mathbb{R}$ in the 18th century were greatly advanced by considering the field extension $\mathbb{C}/\mathbb{R}$, which introduces two principle simplifications:

- An algebraic: it is algebraically closed
- and a geometric: we have the theory of holomorphic functions

This is the first ***insight : base change is fundamental***

> ”The properly geometric view is to abstract from the special properties of the solultions to Problem 1 in the space k, to consider the solutions for any k-algebra, k', and we study how these solutions vary with k'. In particular we are interested in properties that are *stable*.”

Next Grothendieck points out advances were made for Problem 2 over $\mathbb{C}$ by translating them into projective geometry, $\mathbb{C} \cup \{\infty\}$. This is in essence the process of glueing, going from affine to projective.

The second ***insight : gluing is key***.

4.2 The Abstract Variety

So Grothendieck has identified the problems that he sees as being proper to algebraic geometry. He has proposed at least conceptually solutions. We now want to develop some theory in which these solution are easy to deal with.

The first insight leads to the formulation of the abstract variety; functoriality is the property that makes base change easy to talk about. Problem 1 is essentially definining two functors.

Definition (Affine Space)**.**

$$\mathbb{A}^I_k : k - Alg \to Sets$$
$$A \mapsto A^I$$
$$\varphi \mapsto \varphi^I$$

Definition (Variety)**.** *If $S = (F_j)_{j \in J} \in P^J$ as in Problem 1 then we get a subfunctor of affine space*

$$V_S : k - Alg \to Sets$$

$$A \mapsto \{a \in A^I : \forall j \in J \;\; F_j(a) = 0\}$$

this is algebraic Geo

So we can think of algebraic geometry as merely the study of this functor. There are two ways to look at it

- Consider V_S intrinsically, up to isomorphism of functors
- Study it as a subfunctor of affine space, with an embedding

Grothendieck claims that the second is not very interesting

why

4.2.1 Representability

First notice that if $S \subseteq P_I$ then we actually have equality of functors between

$$V_S = V_{(S)}$$

where (S) is the ideal of P_I that is generated by S. So we consider only functors of the form $V_{\mathfrak{J}}$ where $\mathfrak{J}$ is an ideal of P_I.

Theorem. *There is an isomorphism of functors*

$$\mathbb{A}^I_k \cong \mathrm{Hom}_{k-alg}(P_I, -)$$

Proof. The isomorphism on points

$$A^I \to Hom_k(P_I, A^I)$$

is given by $t \mapsto ev_t$.

□

WRITE EVERYTHING

Theorem.

$$V_{\mathfrak{J}} \cong \mathrm{Hom}_k(P_I/\mathfrak{J}, -)$$

Proof. Using that $V_{\mathfrak{J}}$ is a subfunctor we can compose with the iso of functors and see what its image is. It gets mapped to

$$\{f \in Hom_k(P_I, A) : f(\mathfrak{J}) = 0\} \cong \mathrm{Hom}_k(P_I/\mathfrak{J}, A)$$

□

WRITE EVERYTHING

From now on we will conflate the two, if we wish to refer explicitly to the former we will use the notation $V_{\mathfrak{J}}^{\mathbb{A}}$

Thus we have actually shown that every variety is representable. Notice that if $A \in k - Alg$ then there exists some index set I and some ideal $\mathfrak{J} \subseteq P_I$ such that $A = P_I/\mathfrak{J}$.

WRITE EVERYTHING

Hence up to isomorphism all representable functors are varieties.

Thus we can conflate algebraic geometry as merely the study of representable functors.

this is algebraic Geo

4.2.2 Opposite Category

Inclusons $V_A \to \mathbb{A}^I$ are in bijection with (k-algebra) surjections $P_I \to A$.

WRITE EVERYTHING

Surjections of this form are specified by the images of the variables $(T_i)_{i\in I}$ and hence are specified uniquely by an element $(t_i)_{i\in I} \in A^I$.

Theorem. *There is an equivilence of categories between the category of representable functors from k-algebras to sets and the opposite category of k-algebras.*

Proof. The functor sends

$$A \mapsto V_A$$

□

WRITE EVERYTHING. Should follow from Yoneda

So algebraic geometry can be thought of as the pure study of k-algebras.

this is algebraic Geo

4.2.3 Geometric Points

We mirror the classical story of algebraic geometry but now over an arbitrary ring k. First of all let

$$A_{\mathfrak{J}} \quad := \quad P_I/\mathfrak{J}$$

Then if $f \in A_{\mathfrak{J}}$ we can look at its preimages $\pi^{-1}(f)$ under the cannonical surjection $P_I \to A_{\mathfrak{J}}$.

Lemma. *The function*

$$ev_F(B) : V_{\mathfrak{J}}^{\mathbb{A}}(B) \to B$$
$$t \mapsto F(t)$$

is independent of the choice of $F \in \pi^{-1}(f)$. Hence it defines a function on the objects of the two categories Varieties (subfunctors of affine space such that blah) to k-algebras.

Denote by $f_B = ev_F(B)$ for some $F \in \pi^{-1}(f)$.

Proof. □

WRITE EVERYTHING. Does it define a fully fledged functor?

We get a k-algebra homomorphism

$$A \to Hom_k(V_{\mathfrak{J}}^{\mathbb{A}}(B), B)$$

$$f \mapsto f_B$$

which induces the k-homomorphism for every $t \in V_{\mathfrak{J}}^{\mathbb{A}}(B)$

$$ev_t : A \to B$$

$$f \mapsto f_B(t)$$

All of this is to show that we have an inclusion of the points $t \in V_{\mathfrak{J}}^{\mathbb{A}}(B)$ and maps $\mathrm{Hom}_k(A_{\mathfrak{J}}, B)$.

is it a bijection?? Didnt we already know this frmo the equivilences above...

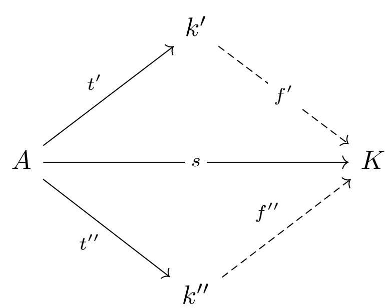
Geometric Points Consider the points of V_A on a k-algebra k' that also happens to be a field. We call these ***geometric points***. We define an equivilence relation on geometric points as follows: The geometric points

$$t' : A \to k', \quad t'' : A \to k''$$

are equivilent if there exists a geometric point $s : A \to K$ as well as two k-homomorphisms

$$f' : k' \to K, \quad f'' : k'' \to K$$

making the diagram commute



Lemma. *This is iff the kernels of t', t'' agree*

Proof. f', f'' are necisarily injective hence commutativity implies that the kernels agree.

is this even true?

For the other direction: Every subring of a field is an integral domian. Thus the kernels are prime ideals in A, call it $\mathfrak{p}$. If they agree then we know that both k', k'' are field extension of

$$FracA/\mathfrak{p}$$

hence there exist cannonical morphisms making the diagram commute.

□

We call the equivilence classes of geometric point ***places***. We have also discovered then that places are in bijection with prime ideals of A.

4.2.4 Spec

We define a functor

$$\text{Spec} : k - \text{Alg} \to \text{Sets}$$

$$A \mapsto \{ \text{ prime ideals of A} \}$$

We give this set a topology as follows Given a subset $S \subseteq A$ we get a subset of $V_A^{\mathbb{A}}(B)$ for every k-algebra B, defined by where f_B annihilates for all $f \in S$. Let $V(S)$ be the places associated to the points where defined above. Then by our bijection these places pick out some collection of prime ideals. We define the Zariski topology on $\text{Spec}\, A$ to be the one generated by these being the closed sets.

whats the field, or does it not matter, stable...

This is not an injection however, we cannot recover A from this topological space.

Let $f \in A$ then define

$$D(f) \quad := \quad \text{Spec}\, A - V(f)$$

this is an open set, and moreover forms a basis for the topology. We define a sheaf of k algebras on $X = \text{Spec}\, A$ (the topological space), denoted $\mathcal{O}_X$ defined on this basis as

$$\mathcal{O}_X(D(f)) \quad := \quad \left\{ \frac{g}{f^n} \in FracA : n \geq 0, \quad g \in A \right\}$$

where by convention $\mathcal{O}_X(D(0)) =$.

Now we can recover A simply as $\mathcal{O}_X(X) \cong A$. Moreover this turns out to be a locally ringed space and hence we have defined the (contravariant) functor

$$\text{Spec} : k - \text{Alg} \to \text{Locally Ringed Spaces}$$

which behaves in such a way that for any $\varphi : A \to B$

$$\Gamma \circ \text{Spec}(\varphi) = \varphi$$

where Γ is the global sections functor.

Theorem. *Spec is fully faithful.*

WRITE EVERYTHING.

The full subcategory that Spec defines are called affine schemes on k.

So now we have shown that algebraic geometry is the study of affine schemes

this is algGeo

4.3 Schemes

The second insight, that gluing is essential, projectivisation, is what forces us to define schemes.

We simply define them as ringed spaces that are locally affine schemes.

locally?

This defines a functor as well, for any scheme X we have

$$X(-) : k - \text{Alg} \to \text{Sets}$$

$$A \mapsto \text{Hom}_k(Spec(A), X)$$

We also have a functor from schemes over k to the functor category of functors from k-algebras to sets, defined by above, and it is fully faithful.

what are the morphisms here? ringed spaces or what...?

WRITE EVERYTHING.

I have heard it said that schemes and stacks are what make certain things representable, they are the objects and categories that allow us to write something or other as Hom(- , -) flesh this out

4.3.1 The Role of k

The information of $\operatorname{Spec} A$ is that of A as a ring and a morphism $k \to A$ to specify the algebra structure. Hence this is the same as a morphism $\operatorname{Spec} A \to \operatorname{Spec} k$. We might then formulate schemes intrinsically and talk about schemes over schemes. An intrinsic scheme is the same as a scheme over $\mathbb{Z}$ because every ring has a unique $\mathbb{Z}$ algebra structure.

5 Stacks

Drawing mainly from [**?**]. Unlike schemes and varieties stacks are a generalisation of these things in the purely categorical direction of representable functors. In fact the notion of stack is purely categorical, it is only with the added data of an Algebraic stack that we get something "geometric" in the sense of relating directly to schemes.

Stacks are a generalisation of schemes and were born out of moduli problem. We want to parametrise geometric things, say schemes, but also give the parametrising object the structure of a scheme. This is not in general possible but with suitable enlargements of the category it often is, this is what motivated stacks.

5.1 As Sheaves

Given a scheme M over another S we have its functor of points

$$Hom_S(-, M)$$

If the category of schemes over S, Sch/S, is given the etale topology then this functor defines a sheaf (this is the notion of space). Thus we can think of schemes as being sheaves of sets on a certain site. this is alg geo

Example. *We have the functor of rank n vector bundles over a fixed scheme X*

$$Bun_n^X : (Sch/X) \to Set$$

$A \mapsto \{$*isomorphism classes of vector bundles over* $X \times A\} = \{$*collections of bundles over* X *parametrised by* $A\}$

This does not form a sheaf becuase it is not representable, it has non-trivial automorphisms. We can sheafify but this makes two bundles equivilent not up to iso but up to local isomorphism (using the etale topology on the representing scheme). So if we want to classify all vector bundles up to isomorphism this wont do.

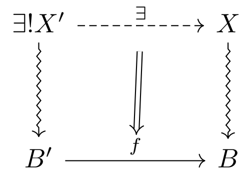
We define the 2-category of ***groupoids*** ($Groupoids$) by

- Objects: Groupoids; categories with all morphisms being iso.
- 1-Morphisms: functors
- 2-Morphisms: natural transformations.

Then a presheaf of groupoids on Sch/S is a contravariant functor into (Groupoids), where morphisms go to functors and diagrams go to natural transformations. A ***stack*** is a sheaf of groupoids on the site Sch/S.

5.2 As Categories

A category over Sch/S is a category F with a functor $p_F : F \to (Sch/S)$. We say that such a category is fibered in groupoids iff for every $f : B' \to B \in Sch/S$ and every $X \in F$ such that $p_F(X) = B$ there is a lift; $\exists!\ X' \in F$ and $\varphi : X' \to X$ such that $p_F(X') = B', p_F(\varphi) = f$. i.e. the following diagram, where vertical lines are application of the functor p_F



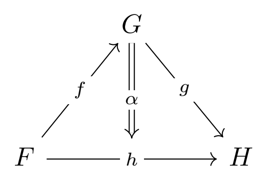
We can connect this to our first definition of stack as follows:

- The map
$$B \mapsto F(B)$$
where $F(B)$ is the category with objects "lying over B", i.e. objects $X \in F$ such that $p_F(X) = B$, and morphisms that lie over the identity on B.
- A presheaf of groupoids G, defines a category by taking objects to be $(B, X) \in (Sch/S) \times G(B)$ and the induced morphisms.

Finally a stack is a category over Sch/S fibered in groupoids satifying the sheaf conditions translated through the above (they are gross).

5.3 Basic Properties

A morphism of stacks is a functor beteween the categories $f : F \to G$ such that $p_G \circ f = p_F$, when f is an equivielence of categories we call it an isomorphism of stacks. A diagram of stacks is a 2-diagram



such that $\alpha : g \circ f \to h$ is an isomorphism of functors.

Scheme to Stack

If we have a scheme $U \in Sch/S$ then the category Sch/U with the functor

$$p_U : (Sch/U) \to (Sch/S)$$

$$(B \to U) \mapsto (B \to U \to S)$$

makes Sch/U into a stack. As a two functor it is simply

$$\mathrm{Hom}_S(-, U)$$

made into a category with only the identity morphisms.

We say that a stack is represented by a scheme when it is isomorphic to such a stack.

Lemma. *If a stack has objects with non-trivial automorphisms then it is not represented by a scheme*

Lemma. *The following is an equivilence of categories: Let F be a stack and U a scheme*

$$\mathrm{Hom}_{stacks}(U, F) \to F(U)$$

$$f \mapsto f(id_U)$$